EIGHTH INTERNATIONAL CARTOGRAPHIC CONFERENCE
USSR, Moscow, august 1976

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The USSR National Committee of Cartographers

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The subject of this paper is the investigation into the problem of producing the projections best possible as to deformation values (with or without limitations on properties).

The problem of producing optimal projections comprises a number of theoretical and practical tasks:

I) Evaluation of the quality of projections, the establishment of criteria of optimal projection.

2) Solving theoretical items on the presence of the optimal withhin the outlined sense projection, on the necessary sufficient criteria of presence, giving the possibility to orientate search for optimal projections.

3) Creation of methods, giving the possibility to operate the properties of projections.

4) Production of optimal projections for the maps of a given territory.

In accordance with the set aim, these questions of evaluation of the quality of projection have been analysed; a mathematical model, giving the possibility to operate the main properties of projections, has been created; the theorem of the presence of the optimal projection according to the value of deformations on a given territory has been proved; necessary criteria of its presence have been determined; concrete variants of projections for the maps of the USSR have been produced.

II. The methodology of the system investigation has been made use of, its main principal items being: I) A precise formulation of the aims of the functioning of a system. 2) The principle of necessary diversity, taking into account all the elements sufficient for the system from the stand point of the aims of its functioning. 3) Choice or construction of criteria of the evaluation of quality and criteria of optimization corresponding to the aims of the functioning of a system. 4) The principle of freedom of choice of solutions under given limitations. 5) Using specific system methods, such as method of models for regulating and designing a system.

The whole set of projections of a given territory of the Earth's surface on to the plane is looked upon by the present author as an object, everything needed to be known of it aimed at producing the optimal projection being looked upon as a system. The properties of cartographic projections are the elements of a system. The surrounding medium appears to be everything that influences the requirements to the mathematical base and the properties of projections i.e. the designation of maps, the peculiarities of their content, the technology of their creating, using, size, shape, character of territory, which is being mapped.

The state of a system, its quality are determined by the properties of projections. Any fixed state represents a certain projection.

Hence, the system under study is an abstract, conceptual and dynamic one. It's notable for its comlexity, integrity and proper arrangement, presence of aims of functioning, caused by the problem of producing optimal projections on definite territories.

III. The properties connected with deformations and the kind of lines of cartographic grid should be refferred to the properties of projections more essential from the point of view of the aims of the functioning of a system.

Every requirement to the projection but a few may be represented by the above mentioned properties of importance. In evaluating the deformations the author suggests to differentiate the following properties: the value of deformations, the character of deformations, the peculiarity of deformation changes and the peculiarity of deformation distribution over a cartographic territory under mapping.

The values of deformation are characterized by 3 levels of criteria: I) deformations at a point in a given direction; 2) deformations at a point, 3) deformations within the limits of the territory under mapping.

The criteria of values of deformations of any level may be simple, determining the deformation of a certain kind (length, angles or areas) or complex, characterizing the deformations of various kinds.

Optimization according to simple criteria, for instance, according to the deformation of area leads to trivial solutions, i.e. to projections free from corresponding deformation. However, practice puts forward requirements to projections, stimulating the optimization according to several criteria. It is suggested in this paper to evaluate the value of deformations by means of complex criteria.

Let's take a vector $\bar{\rho}=\{p-1, w-1\}$, for which the deformations of area $p-1$ and forms $w-1=(a/b)-1$, where $a-1$, $b-1$ - are extremal deformations of length - are its projections. It is suggested to take the length of this vector

$$\bar{\rho}=\sqrt{(p-1)^2+(w-1)^2} \qquad /1/$$

for the complex characteristic of deformation at a point. Variational or minimax criteria, whose cores constitute the criterion ρ i.e.

$$E_1=\frac{1}{\mathcal{F}}\int_{\mathcal{F}}\rho^2 d\mathcal{F}, \qquad /2/$$

$$E_2=(\rho_2-\rho_1)=\max_{\mathcal{F}}\Delta\rho, \qquad /3/$$

where ρ_2, ρ_1 - the greatest and the least values of ρ, may be taken as criteria of deformations within the limits of cartographic territory $\mathcal{F}$.

Criteria /1/,/2/,/3/ may be generalized, introducing weights into the coordinates of a vector: $\bar{\rho}=\{\ell_1(p-1); \ell_2(w-1)\}$.

The criteria suggested are well - grounded, effective and of real sense. They generalize the applied criteria I,2,3 and don't violate the existing traditions of the evaluation of projections; as in case of conformal projection - they lead to the value of area deformation, in case of equivalent ones they cause the deformation of shape. They are applicable to the evaluation of deformations of any projection.

The character of deformations is a property, up to nowadays determined by intuition, without any quantitative evaluation.

Investigations showed that the character of deformations may be determined as the correlation of deformations of various kinds. The value of angle α, composed of vector $\bar{\rho}$ of any projection with vector $\bar{K}$ of conformal projection may be introduced as the measure of the character of deformation.

Really, vectors of conformal $\bar{K}=\{p-1, 0\}$ and equivalent $\bar{e}=\{0, w-1\}$ projections are orthogonal $(\bar{K}\cdot\bar{e}=0)$, independent and constitute basis. Angle α, for which

$$\cos\alpha=\frac{p-1}{\rho}, \quad \sin\alpha=\frac{w-1}{\rho}, \quad tg\,\alpha=\frac{w-1}{p-1},$$

gives the possibility to classify projections according to this criterion.

The character of deformations and, as well as the value of complex deformation are generalized local characteristics, which change while transgressing from point to point but still remaining independent of direction. The conformal, equivalent and equidistant projections are of permanent character of deformations equal to 0, $\pi/2$ and $\pi/4$ respectively. Besides, there is an infinite set of projections with permanent character of deformation, the latter varying in value.

If, for example, the equation of deformations $\rho=\rho(\varphi,\lambda)$ is understood as a certain surface over the territory being mapped, then the type of this surface, its "relief", the shape of its lines of equal deformation, the distribution of points and lines of extremal values of the deformation (central points and lines) and other structural lines of the surface of deformation characterize the peculiarity of the change in the deformation.

The peculiarity of distribution of deformations in any projection is determined by detailed redistribution of deformations over the territory $\mathcal{F}$ under mapping and may be expressed by some input data, e.g. by giving values for deformations at the central point or on the central line, on the boundary line or on any other curve, as well as disposition of the standard lines free from deformations along them.

The kind of a cartographic grid is characterized by the type of its lines, their mutual disposition, curvature and particularities of representation of poles.

The curvature of lines at a point or the mean curvature are usually evaluated visually or calculated by means of formulas, if necessary. To produce optimal projections we have proposed to express the curvature of the lines of cartographic grids via convergence of meridians γ and deviation of the angle between meridians and parallels ε from the right angle

$$K_M = -\frac{\gamma_\varphi}{mM}, \quad K_n = \frac{\gamma_\lambda - \varepsilon_\lambda}{2n},$$

where K_M, K_n – the curvatures of meridians and parallels in the projection, respectively; m, n – the scales of length along meridians and parallels; M, z – the radii of curvature of meridians and parallels of the Earth's surface; $\gamma_\varphi, \gamma_\lambda, \varepsilon_\lambda$ – denotations of corresponding derivatives.

The type of lines may also be characterized by curvature: if the lines are straight, then $K=0$, in case of circles K = const. etc.

Mutual disposition of lines of the cartographic grid can be

determined by denotory of the angle θ between meridians and parallels or its deviation from the right angle ε.

Hence, characteristics γ and ε or K_M and K_n determine first and foremost the kind of the cartographic grid, meanwhile the characteristics ρ, α, as well as the "input data", symbolized as Δ, influence the deformations in the projections. The principal system of characteristics, suggested by the present writer, includes: the value of complex deformation ρ, the character of the deformation α, convergence ~~of approximation~~ of the meridians γ, deviation of the angle between meridians and parallels from the right angle ε and the "input data" Δ, closely related to the territory $\mathcal{F}$ under mapping. Other characteristics are considered to be auxiliary and complementary to the principal set of characteristics, which they are expressed through.

Taking into account only the values of deformations, we take minimax criterion /3/ for the criterion of optimization.

IV. One of the methods of operating and designing any system is that of models. The author thinks there are four types of models to produce projections: a) models of the perspective of the Earth's surface on to a plane, cylinder, cone; b) direct analytical models, determining first and foremost the kind of the cartographic grid on the whole; c) inverse analytical models, determining the relationship of local characteristics of projections; d) graphic models as drafts of cartographic grids.

Each type of models operates its own method of producing projections. It is better to use an inverse analitical model to operate local properties. We have worked out a structural and a local mathematical model, based on the above mentioned principal set of characteristics. (scheme I, formulas 6, 7).

The structural model of the system of cartographic projections is an indicated one, of zero type. It reflects the presence of elements, their interconnections and disaggregation of the system into subsystems and levels by adding one of the main properties of projections at each level. Every level and its subsystems may by regarded as independent systems.

The structural model thus created is mot a classification scheme, but represents one of the possible "technological" schemes to produce any projection by means of a proposed local mathematical model with principal characteristics $\rho, \alpha, \gamma, \varepsilon, \Delta$.

The mathematical model (6, 7) is an incomplete system of equations of coordinates and two differential quasilinear Euler–Urmaev equations in partial derivatives with four functions $\rho, \alpha, \gamma, \varepsilon$ without "input data", i.e. initial or boundary curve and two functions along this curve. The redundant functions in this model i.e. the two characteristics of properties of the projections are considered by the author as regulating ones and the absence of "input data" is offered to be seen as an additional opportunity to operate the properties of projections. Thus, when operating the properties of projections with the help of this model we have five degrees of freedom.

The investigations into the models show that optimization with the help of the E_2 criterion is possible at every level, in every subsystem by establishing the type of redundant functions and "input data".

The common solutions of optimization problems correspond to the optimization at the fourth level and partially at the third one. For example, the P.L. Chebyshev problem on the best conformal projection for a given area corresponds to the optimization in the "a" system of the third level ($\varepsilon=0, \alpha=0$); the A.A. Markov problem on the best cone projection for the territory between two parallels corresponds to the optimization in the „δ" system of the third level ($\gamma=\kappa\lambda+b, \ \varepsilon=0$).

The higher is the level, the greater the number of degrees of freedom, the more complex is optimization, and it is possible only under limiting conditions, created by determining certain characteristics. These characteristics should be coordinated with the main objective of the functioning of the system. It is necessary to analyse their influence on the values of deformations and other properties of projections.

V. Using the model in question, we have analysed the influence of the characteristics $\alpha, \gamma, \varepsilon, \Delta$ on the value of deformations and other properties of projections. The principal results of this investigation are as follows:

I. The conformal $(\alpha=0)$ and equivalent $(\alpha=\pi/2)$ projections have the same value for criterion E_2 within the boundaries of the defined territory, other conditions being equal.

2. The value of criterion E_2 will be less, if $\alpha\to\pi/4$ (other conditions being equal).

3. The change but the character of deformations may result in smaller variations of ρ, compared with changes of "input data" and γ. For instance, an E_2 criterion comparison of equidistant $(\alpha=\pi/4)$, conical and oblique cylindrical projections $(E_2\approx 5\%)$ with the N.L. Chebyshev conformal projection $(E_2\approx 7\%)$ for the USSR maps reveals the advantage of equidistant projections.

4. The Euler–Urmaev type system (7) related with "input data" arrangement is dependent on the characteristics $\alpha, \gamma, \varepsilon$. In case of orthogonal projections the type of this system depends chiefly on α, i.e., if $\alpha<\pi/4$ the system is of elliptical type, while under $\alpha>\pi/4$ it is hyperbolic.

5. In case of conformal projections $(\alpha=0)$ the Euler–Urmaev system turns into a linear one, for which the Dirichlet problem is solved in the presence of any geometrical shape of the territory that is homeomorphic to a circle. Under $\alpha<\pi/4$ the system will be a quasilinear one, for which even in case of a convex contour of the territory, the Dirichlet problem is not always soluble. Even this only thing prevents us from thinking, that within a given territory the best projections for the value of deformations, which are described by a quasilinear system of elliptical type, will be those, possessing a constant boundary deformation [3].

6. Among orthogonal equivalent projections $(\varepsilon=0, \alpha=\pi/2)$ only direct azimuthal ones possess closed circular lines of equal deformation ρ. No other orthogonal equivalent projections, possessing circular and oval lines of equal deformations ρ, exist.

7. The set of orthogonal projections $(\varepsilon=0)$, for which $\alpha=\alpha(\varphi), \gamma=\gamma(\lambda)$ has rectilinear meridians. The least complex deformations in this set have conical, cylindrical and azimuthal projections $(\gamma=\kappa\lambda+b)$.

The orthogonal projections, for which $\gamma=\gamma(\varphi)\neq 0$ summarize Korkin-Grave projections and possess considerable complex values of deformations.

8. The presence of the best projection is perceived usually by intuition. It is shown, that the presence of the best projection, possessing minimal E_2 in the closed bounded territory in

the class of twice continuously differentiable functions, is followed out of the Weierstrass theorem: any continuous function given on a compact set reaches its sup. and inf.

Actually, all the functions $\rho, \alpha, \gamma, \varepsilon$ of models (6, 7) are twice continuously differentiable (according to the conditions of greated model) Function

$$E_2(\alpha, \gamma, \varepsilon) = \rho_2(\alpha, \gamma, \varepsilon, \mathcal{F}) - \rho_1(\alpha, \gamma, \varepsilon, \mathcal{F})$$

may be interpreted as produced out of ρ by means of fixing values $\varphi, \lambda \in \mathcal{F}$. Consequently E_2 will be a continuous function of $\alpha, \gamma, \varepsilon$. In consequence of boundedness of $\alpha, \gamma, \varepsilon$ $(-\pi \leq \alpha < \pi, 0 \leq \gamma < 2\pi, 0 \leq \varepsilon < \pi)$ in the limited closed territory $\mathcal{F}$ the domain of their values, i.e. the domain of given data of the function E_2 will be closed, founded, i.e. compactum. Thus, the function E_2, determining the projections, is continuous on the compactum and according to the Weierstrass theorem assumes the least value on it, i.e. $\min\limits_{\alpha,\gamma,\varepsilon} \max\limits_{\mathcal{F}} \Delta\rho$.

9. The necessary conditions of the presence of the best projection (in the sense of minimum of the greatest variation of complex deformation ρ within the limits of the given $\mathcal{F}$ terrotory) are the arrangement of the central points or lines of complex deformation ρ in the middle part of the territory under mapping, meanwhile the extremal values of ρ are to be on the boundary and in the centre. These necessary premises are weak, but they orientate for projections search, best in the sense of minimal value of E_2 [2].

VI. These investigations permited to outline the following methods of producing the best projections:

1. The aim, which the projection is created for, is being clearly formulated.
2. With this view the following set of requirements on the projections is being established;
3. The requirements are being formalized and represented in characteristics of the properties of projections.
4. In accordance with the aim of the functioning of the system the optimization criterion, the guiding parameters and limitations are being established. The limitations are represented by such values of some characteristics and "input data", under which necessary conditions of the presence of the

optimal projections are being realized.

5. The mathematical model of projection to the desribed aim is either being selected or created.
6. The problem is being mathematically formulated.
7. The projection is being determined directly by means of the selected model or with the help of successive approximation.
8. Analysis of agreement of a produced projection with requirements to it and comparison with actual variants is being fulfilled.

VII. The suggested methods were applied to produce a set of projections for the USSR territory maps (and other regions).

The best distinguished by minimality of E_2 is an equidistant conic projection, with complex deformations being equal at the extreme parallels. In case of necessity to be drawn on the map of pole the best projection will be that of oblique, equidistant, of a cylindrical type.

To serve special tasks of conformal projections are recommended that by N.L. Chebyshev and a conformal conical one. Of the projections with small values of deformations of angles a conical projection with $\alpha = 15°$ is recommended, meanwhile of the projections with small deformations of areas a conical projection with $\alpha = 75°$ is recommended. The polyconic projection with a changing character of deformation and with central point in the middle part of the territory of the USSR is of interest.

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$$x=\int_{\varphi_0}^{\varphi}\mu(\varphi,\lambda)\cos\gamma(\varphi,\lambda)\,d\varphi-\int_{\lambda_0}^{\lambda}\nu(\varphi_0,\lambda)\sin[\varepsilon(\varphi_0,\lambda)-\gamma(\varphi_0,\lambda)]\,d\lambda,$$

$$y=-\int_{\varphi_0}^{\varphi}\mu(\varphi,\lambda)\sin\gamma(\varphi,\lambda)\,d\varphi+\int_{\lambda_0}^{\lambda}\nu(\varphi_0,\lambda)\cos[\varepsilon(\varphi_0,\lambda)-\gamma(\varphi_0,\lambda)]\,d\lambda; \qquad (6)$$

$m > n$

$$r a_- \rho_\varphi + r\rho b_+ \alpha_\varphi - rpwc_- tg\,\varepsilon\,\varepsilon_\varphi - 2rpwh\,tg\,\varepsilon\,\gamma_\varphi -$$
$$- Mpc_+ h\gamma_\lambda + 2Mpwh\sin\varphi = 0,$$

$$(M/2w)a_+c_+\cos\varepsilon\,\rho_\lambda + (M\rho/2w)\,b_-c_+\cos\varepsilon\,\alpha_\lambda - 2Mpwh\sec\varepsilon\,tg\,\varepsilon\,\varepsilon_\lambda -$$
$$- Mpc_+h\sin\varepsilon\,\gamma_\lambda + 2rpwh\sec\varepsilon\,(\varepsilon_\varphi-\gamma_\varphi) = 0; \qquad (7)$$

$m < n$

$$(r/2w)a_+c_+\cos\varepsilon\,\rho_\varphi + (r\rho/2w)\,b_-c_+\cos\varepsilon\,\alpha_\varphi - 2rpw^2\sec\varepsilon\,tg\,\varepsilon\,\varepsilon_\varphi -$$
$$- rpc_+h\sin\varepsilon\,(\varepsilon_\varphi-\gamma_\varphi) + 2Mpwh\sec\varepsilon\,\gamma_\lambda - Mpc_+h\sin\varphi = 0,$$

$$Ma_-\rho_\lambda + M\rho b_+\alpha_\lambda - Mpwc_+ tg\,\varepsilon\,\varepsilon_\lambda - rpc_+h(\varepsilon_\varphi-\gamma_\varphi) + 2Mpwh\,tg\,\varepsilon\,\gamma_\lambda = 0; \qquad (7a)$$

where

$$a_\pm = p(w^2-1)\sin\alpha \pm wh\cos\alpha; \quad b_\pm = p(w^2-1)\cos\alpha \pm wh\sin\alpha;$$

$$c_\pm = w^2+1\pm h; \quad h=\sqrt{(w^2+1)^2-4w^2\sec^2\varepsilon} = wAB;$$

$$A=\sqrt{w^2+1+2w\sec\varepsilon}/\sqrt{w}; \quad B=\sqrt{w^2+1-2w\sec\varepsilon}/\sqrt{w};$$

$$p = 1+\rho\cos\alpha; \quad w = 1+\rho\sin\alpha; \quad \mu = Mm; \quad \nu = rn;$$

$$m=(\sqrt{p}/2)(A-B); \quad n=(\sqrt{p}/2)(A+B); \quad (m<n)$$

$$m=(\sqrt{p}/2)(A+B); \quad n=(\sqrt{p}/2)(A-B); \quad (m>n)$$

$\underline{\varepsilon = 0}$

$$x=\int_{\varphi_0}^{\varphi}\mu\cos\gamma\,d\varphi+\int_{\lambda_0}^{\lambda}(\nu\sin\gamma)_{\varphi=\varphi_0}\,d\lambda; \quad y=-\int_{\varphi_0}^{\varphi}\mu\sin\gamma\,d\varphi+\int_{\lambda_0}^{\lambda}(\nu\cos\gamma)_{\varphi=\varphi_0}\,d\lambda;$$

$m < n$

$$(\sin\alpha-\cos\alpha)\rho_\lambda + \rho(\sin\alpha+\cos\alpha+\rho)\alpha_\lambda + 2(r/M)(1+\rho\cos\alpha)(1+\rho\sin\alpha)^2\gamma_\varphi = 0;$$

$$(\sin\alpha+\cos\alpha+\rho\sin 2\alpha)\rho_\varphi + \rho(\cos\alpha-\sin\alpha+\rho\cos 2\alpha)\alpha_\varphi +$$
$$+ 2(M/r)(1+\rho\cos\alpha)[\gamma_\lambda-(1+\rho\sin\alpha)\sin\varphi] = 0;$$

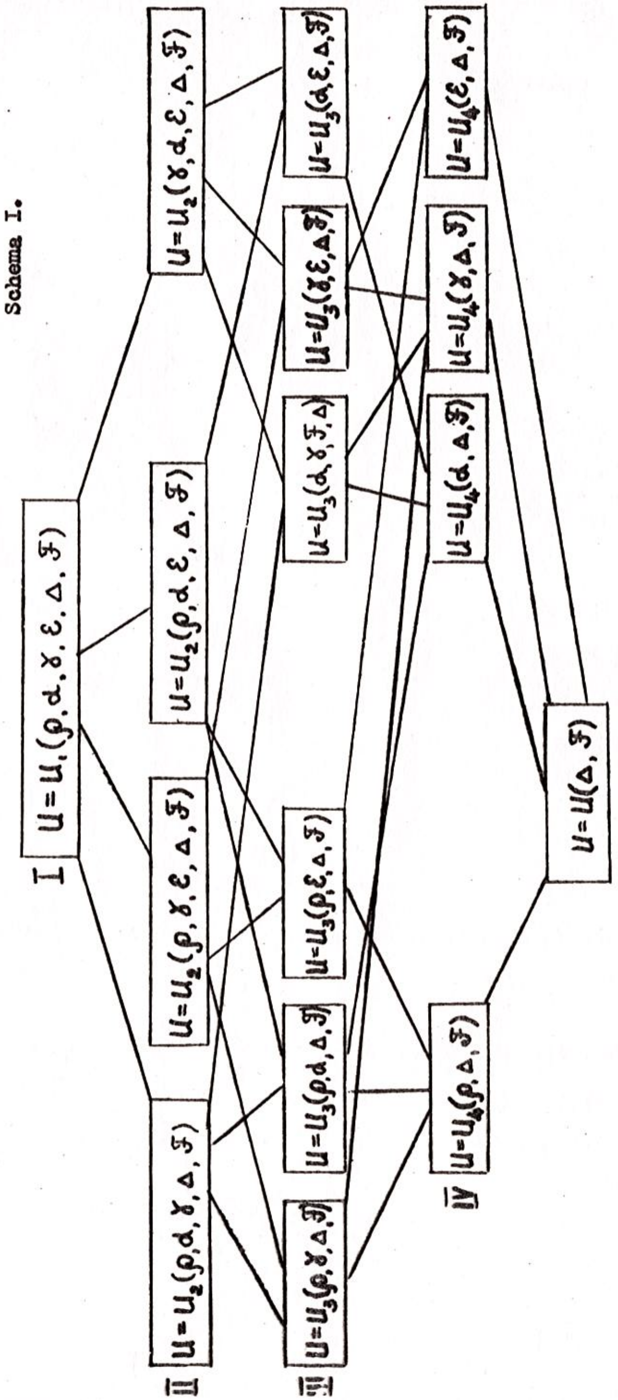
$m > n$

$$(\sin\alpha+\cos\alpha+\rho\sin 2\alpha)\rho_\lambda + \rho(\cos\alpha-\sin\alpha+\rho\cos 2\alpha)\alpha_\lambda -$$
$$- 2(r/M)(1+\rho\cos\alpha)\gamma_\varphi = 0;$$

$$(\sin\alpha-\cos\alpha)\rho_\varphi + \rho(\sin\alpha+\cos\alpha+\rho)\alpha_\varphi -$$
$$-2(M/r)(1+\rho\cos\alpha)(1+\rho\sin\alpha)[(1+\rho\sin\alpha)\gamma_\lambda-\sin\varphi] = 0.$$

Schema I.

Structural model of mapping projections system



The Euler-Urmaev system $\Psi_i(\rho_{\mathcal F}, \alpha_{\mathcal F}, \gamma_{\mathcal F}, \varepsilon_{\mathcal F}, \mathcal{F}) = 0$

Denotations: $U_i = \binom{x_i}{y_i}$, $\varphi,\lambda \in \mathcal{F}$; $\rho_{\mathcal F} = \binom{\rho_\varphi}{\rho_\lambda}$, $\alpha_{\mathcal F} = \binom{\alpha_\varphi}{\alpha_\lambda}$, $i = 1,2$